CHARACTERISTIC RANK OF VECTOR BUNDLES OVER STIEFEL MANIFOLDS

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ABSTRACT. The characteristic rank of a vector bundle ξ over a finite connected CW-complex X is by definition the largest integer k, $0 \leq k \leq \dim(X)$, such that every cohomology class $x \in H^j(X;\mathbb{Z}_2)$, $0 \leq j \leq k$, is a polynomial in the Stiefel-Whitney classes $w_i(\xi)$. In this note we compute the characteristic rank of vector bundles over the Stiefel manifold $V_k(\mathbb{F}^n)$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

1. INTRODUCTION

Let X be a connected finite CW-complex and ξ a real vector bundle over X. Recall [5] that the *characteristic rank of* ξ over X, denoted by $\operatorname{charrank}_X(\xi)$, is by definition the largest integer k, $0 \leq k \leq \dim(X)$, such that every cohomology class $x \in H^j(X;\mathbb{Z}_2)$, $0 \leq j \leq k$, is a polynomial in the Stiefel-Whitney classes $w_i(\xi)$. The *upper characteristic rank of* X, denoted by $\operatorname{ucharrank}(X)$, is the maximum of $\operatorname{charrank}_X(\xi)$ as ξ varies over all vector bundles over X.

Note that if X and Y are homotopically equivalent connected closed manifolds, then $\operatorname{ucharrank}(X) = \operatorname{ucharrank}(Y)$. When X is a connected closed smooth manifold and TX the tangent bundle of X, then $\operatorname{charrank}_X(TX)$, denoted by $\operatorname{charrank}(X)$, is called the *characteristic rank of the manifold* X (see [3]).

The characteristic rank of vector bundles can be used to obtain bounds for the $\mathbb{Z}_2$-cup-length of manifolds (see [1],[3] and [5]). In some situations, the value of the upper characteristic rank can be used to show the vanishing of the Stiefel-Whitney class of a certain degree for all vector bundles. An important task is therefore to understand the characteristic rank of vector bundles.

In [5], the second and third named authors have computed the characteristic rank of vector bundles over: a product of spheres, the real and complex projective spaces, the Dold manifold $P(m, n)$, the Moore space $M(\mathbb{Z}_2, n)$, and the stunted projective space $\mathbb{RP}^n/\mathbb{RP}^m$.

Let $\mathbb{F}$ denote either the field $\mathbb{R}$ of reals, the field $\mathbb{C}$ of complex numbers or the skew-field $\mathbb{H}$ of quaternions. Let $V_k(\mathbb{F}^n)$ denote the Stiefel manifold of orthonormal k-frames in $\mathbb{F}^n$. In this note we compute the characteristic rank of vector bundles over $V_k(\mathbb{F}^n)$. Our methods are elementary and make use of some well-known facts about the Stiefel manifolds. We prove the following.

Theorem 1.1. *Let* $X = V_k(\mathbb{F}^n)$ *with* $1 < k < n$ *when* $\mathbb{F} = \mathbb{R}$ *and* $1 < k \leq n$ *when* $\mathbb{F} = \mathbb{C}, \mathbb{H}$.

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(1) *If* $\mathbb{F} = \mathbb{R}$*, then*

$$\operatorname{ucharrank}(X) = \begin{cases} n-k-1 & \text{if } n-k \neq 1,2,4,8, \\ 2 & \text{if } n-k = 1 \text{ and } n \geq 4, \\ 2 & \text{if } n-k = 2, \\ 4 & \text{if } n-k = 4 \text{ and } k = 2. \end{cases}$$

(2) *If* $\mathbb{F} = \mathbb{R}$*,* $k > 2$ *and* $n-k = 4$*, then* $\operatorname{ucharrank}(X) \leq 4$.
(3) *If* $\mathbb{F} = \mathbb{R}$ *and* $n-k = 8$*, then* $\operatorname{ucharrank}(X) \leq 8$.
(4) *If* $\mathbb{F} = \mathbb{C}$*, then*

$$\operatorname{ucharrank}(X) = \begin{cases} 2 & \text{if } k = n, \\ 2(n-k) & \text{if } k < n. \end{cases}$$

(5) *If* $\mathbb{F} = \mathbb{H}$*, then* $\operatorname{ucharrank}(X) = 4(n-k) + 2$.

When $\mathbb{F} = \mathbb{R}$ and $n-k = 4, 8$, we only give a bound.

The characteristic rank of vector bundles over $V_1(\mathbb{F}^n)$, which is a sphere, and $SO(3) = \mathbb{RP}^3$ has been described in [5]. Note that $V_n(\mathbb{R}^n) = O(n)$ is not connected. This explains the restrictions on the range of k and the condition $n \geq 4$ in the second equality of (1) in the above theorem.

Notations. The characteristic rank of a vector bundle ξ over X will simply be denoted by $\operatorname{charrank}(\xi)$; the space X will usually be clear from the context. For a space X, $H^*(X)$ will denote cohomology with $\mathbb{Z}_2$-coefficients.

2. Proof of Theorem 1.1

We begin by recalling certain standard facts about the Stiefel manifolds that are needed to prove the main theorem. One fact about Stiefel manifolds that we shall need is a description of the $\mathbb{Z}_2$-cohomology ring of $V_k(\mathbb{F}^n)$. We note this below.

Theorem 2.1. ([2], Propositions 9.1 and 10.3) *We have* $H^i(V_k(\mathbb{F}^n)) = 0$ *for* $i = 1, 2, \ldots, c(n-k+1)-2$ *and* $H^{c(n-k+1)-1}(V_k(\mathbb{F}^n)) \cong \mathbb{Z}_2$*, where* $c = \dim_{\mathbb{R}} \mathbb{F}$*. Further, when* $\mathbb{F} = \mathbb{R}$*, the cohomology ring* $H^*(V_k(\mathbb{R}^n))$ *has a simple system of generators* $a_{n-k}, a_{n-k+1}, \ldots, a_{n-1}$ $(a_i \in H^i(V_k(\mathbb{R}^n)))$ *such that* $a_i^2 = a_{2i}$ *if* $2i \leq n-1$ *and* $a_i^2 = 0$ *otherwise.* □

The action of the Steenrod squares on $H^*(V_k(\mathbb{R}^n)$ is given by (see [2], Remarque 2 in §10)

$$Sq^i(a_j) = \begin{cases} \dbinom{j}{i} a_{j+i} & \text{if } j+i \leq n-1, \\ \\ 0 & \text{otherwise.} \end{cases}$$

For $k \geq 2$ consider the sphere bundle $S^{n-k} \overset{i}{\hookrightarrow} V_k(\mathbb{R}^n) \xrightarrow{p} V_{k-1}(\mathbb{R}^n)$, where p maps a k-frame to the $(k-1)$-frame determined by ignoring the last vector. It is clear, by the above theorem, that the Serre spectral sequence of this sphere bundle is trivial and hence, the homomorphism $i^* : H^{n-k}(V_k(\mathbb{R}^n)) \longrightarrow H^{n-k}(S^{n-k})$ is an isomorphism.

Lemma 2.2. *Provided that* $k \neq n$ *for* $\mathbb{F} = \mathbb{C}$*, let* n *and* k *be as in Theorem 1.1. Then, for any vector bundle* ξ *over* $V_k(\mathbb{F}^n)$*, we have*

(1) $\operatorname{charrank}(\xi) \geq c(n-k+1) - 2$,
(2) $\operatorname{charrank}(\xi) = c(n-k+1) - 2$*, if* $\mathbb{F}$ *is* $\mathbb{C}$ *or* $\mathbb{H}$*,*

(3) $\operatorname{charrank}(\xi) \leq n-k$ *if* $\mathbb{F}$ *is* $\mathbb{R}$ *and* $n-k$ *is even.*

Proof. (1) follows from the cohomology structure of $V_k(\mathbb{F}^n)$. Next observe that by Wu's formula
$$w_{c(n-k+1)-1}(\xi) = w_1(\xi)w_{c(n-k+1)-2}(\xi) + Sq^1(w_{c(n-k+1)-2}(\xi)) = 0.$$
This proves (2). We now come to the proof of (3). Note that, by Theorem 2.1, $H^{n-k}(V_k(\mathbb{R}^n))) \cong \mathbb{Z}_2$ is generated by a_{n-k} and $H^{n-k+1}(V_k(\mathbb{R}^n)) \cong \mathbb{Z}_2$ is generated by a_{n-k+1}. Suppose that $\operatorname{charrank}(\xi) \geq n-k+1$. Then $w_{n-k}(\xi) = a_{n-k}$ and $w_{n-k+1}(\xi) = a_{n-k+1}$. Now by Wu's formula we have
$$w_{n-k+1}(\xi) = w_1(\xi)w_{n-k}(\xi) + Sq^1(w_{n-k}(\xi)) = (n-k)a_{n-k+1} = 0.$$
This contradiction proves (3). □

We are now in a position to prove our main theorem.

Proof of Theorem 1.1. If ξ is a vector bundle over $V_k(\mathbb{R}^n)$ with $w_{n-k}(\xi) \neq 0$, then $i^*\xi$ is a vector bundle over S^{n-k} with $w_{n-k}(i^*\xi) \neq 0$. By Theorem 1 in [4], this is possible only if $n-k = 1, 2, 4, 8$. This and Lemma 2.2(1) prove the first equality in Theorem 1.1(1).

To prove the second equality in Theorem 1.1(1), we note that $H^1(V_{n-1}(\mathbb{R}^n)) = H^1(SO(n))$ is generated by a_1, $H^2(SO(n))$ is generated by $a_2 = a_1^2$ and $H^3(SO(n))$ is generated by $a_1^3 = a_1a_2$ and a_3. Now if ξ is a non-orientable line bundle over $SO(n)$, then clearly $\operatorname{charrank}(\xi) \geq 2$. Now assume that ξ is a vector bundle over $SO(n)$ with $\operatorname{charrrank}(\xi) \geq 3$. Then $w_1(\xi) = a_1$, $w_2(\xi) = ka_2 = ka_1^2$ with $k \in \{0,1\}$ and $w_3(\xi) = a_3$ is not a multiple of a_1^3. But by Wu's formula
$$w_3(\xi) = w_1(\xi)w_2(\xi) + Sq^1(w_2(\xi)) = a_1^3.$$
This contradiction completes the proof.

To prove the third equality in Theorem 1.1(1), we note that by Lemma 2.2(3), we have $\operatorname{charrank}(\xi) \leq 2$ for any vector bundle ξ over $V_k(\mathbb{R}^n)$ when $n-k = 2$. It is well known that $H^2(V_{n-2}(\mathbb{R}^n);\mathbb{Z}) \cong \mathbb{Z}$. Then clearly there is a 2-plane bundle ξ over $V_{n-2}(\mathbb{R}^n)$ with Euler class $e(\xi)$ a generator and hence $w_2(\xi) \neq 0$. This implies that $\operatorname{charrank}(\xi) = 2$. This completes the proof.

In view of Lemma 2.2(3), the proof of the fourth equality in Theorem 1.1(1) will be complete if we exhibit a vector bundle ξ over $V_2(\mathbb{R}^6)$ with $w_4(\xi) \neq 0$. To construct such a bundle, we start with the well-known circle bundle $p : V_2(\mathbb{R}^6) \longrightarrow G_2(\mathbb{C}^4)$. Here $G_2(\mathbb{C}^4)$ denotes the complex Grassmann manifold of complex 2-planes in $\mathbb{C}^4$. Since $H^3(G_2(\mathbb{C}^4)) = 0$, the Gysin sequence
$$\cdots \to H^2(G_2(\mathbb{C}^4)) \xrightarrow{\psi} H^4(G_2(\mathbb{C}^4)) \xrightarrow{p^*} H^4(V_2(\mathbb{R}^6)) \longrightarrow H^3(G_2(\mathbb{C}^4)) \to \cdots$$
of the circle bundle p shows that the homomorphism $p^* : H^4(G_2(\mathbb{C}^4)) \longrightarrow H^4(V_2(\mathbb{R}^6))$ is onto. Let γ be the canonical complex 2-plane bundle over $G_2(\mathbb{C}^4)$ and $\gamma_{\mathbb{R}}$ its underlying real bundle. It is known that $H^2(G_2(\mathbb{C}^4)) \cong \mathbb{Z}_2$ is generated by $w_2(\gamma_{\mathbb{R}})$ and $H^4(G_2(\mathbb{C}^4)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by $w_2^2(\gamma_{\mathbb{R}})$ and $w_4(\gamma_{\mathbb{R}})$. Since $\psi(x) = x \smile (w_2(\gamma_{\mathbb{R}}))$ and $w_2^2(\gamma_{\mathbb{R}}) \neq w_4(\gamma_{\mathbb{R}})$, it is clear that $w_4(\gamma_{\mathbb{R}}) \notin \operatorname{Im}(\psi) = \operatorname{Ker}(p^*)$, thus $p^*(w_4(\gamma_{\mathbb{R}})) = w_4(p^*\gamma_{\mathbb{R}}) \neq 0$. Thus $\xi = p^*\gamma_{\mathbb{R}}$ is the required vector bundle.

The assertions in (2) and (3) follow from Lemma 2.2(3).

To prove the assertion (4) in Theorem 1.1, first assume that $k = n$ with $n \geq 2$. Then, by Theorem 2.1, $H^1(U(n)) \cong \mathbb{Z}_2$. Thus there exists a non-trivial line bundle ξ such that $w_1(\xi)$ generates $H^1(U(n))$. Since $H^2(U(n)) = 0$, it follows that $\operatorname{ucharrank}(U(n)) \geq 2$. Now if

there exists a vector bundle ξ with $w_3(\xi) \neq 0$, then $w_3(\xi)$ generates $H^3(U(n)) \cong \mathbb{Z}_2$. But by Wu's formula we have

$$w_3(\xi) = w_1(\xi)w_2(\xi) + Sq^1(w_2(\xi)) = 0.$$

This is a contradiction. The case $k < n$ follows from Lemma 2.2(2).

The assertion (5) also follows from Lemma 2.2(2). This completes the proof of the theorem. □

We have the following immediate corollary of Theorem 1.1.

Corollary 2.3. (1) *If $n - k \neq 1, 2, 4, 8$ and $1 < k < n$, then we have $w_{n-k}(\xi) = 0$ for any vector bundle ξ over $V_k(\mathbb{R}^n)$.*
(2) *For any non-orientable vector bundle ξ over $SO(n)$, $n \geq 4$, we have either $w_3(\xi) = 0$ or $w_3(\xi) = a_1^3$, where a_1 is the (unique) non-zero element in the first cohomology.*
(3) *Let ξ be a vector bundle over $V_k(\mathbb{R}^n)$, where $n - k$ is even and $1 < k < n$. Then $w_{n-k+1}(\xi) = 0$.*
(4) *For any vector bundle ξ over $U(n)$, we have $w_3(\xi) = 0$.*

□

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